ΜΑΘΗΜΑΤΙΚΑ 10/10/2020

ΘΕΜΑ Α

A4 I) Ε II) Γ III) Γ

ΘΕΜΑ Β

B1 $A_{f\circ g} = \{x \geq 0 \,/\, g(x) \in \mathbb{R}\} = [0,+\infty)$

$$f(g(x)) = \frac{3}{(\sqrt{x})^2+3} = \frac{3}{x+3}, \quad x \geq 0$$

B2 $h'(x) = -\dfrac{3}{(x+3)^2} < 0$ άρα $\downarrow$ στο $[0,+\infty)$ οπότε κ' 1-1

- $h(x) = y \iff \dfrac{3}{x+3} = y \iff 3 = xy + 3y \iff 3 - 3y = xy$

$x = \dfrac{3-3y}{y}$, $y \neq 0$ επίσης $x \geq 0 \iff \dfrac{3-3y}{y} \geq 0 \iff$

$\iff (1-y)y \geq 0$

y		0		1	
	$-$	ϕ ‖	$+$	ϕ	$-$

άρα $y \in (0,1]$

οπότε $h^{-1}(x) = \dfrac{3}{x} - 3$, $x \in (0,1]$

B3 $f'(x) = -\dfrac{3}{(x^2+3)^2} \cdot 2x = \dfrac{-6x}{(x^2+3)^2}$

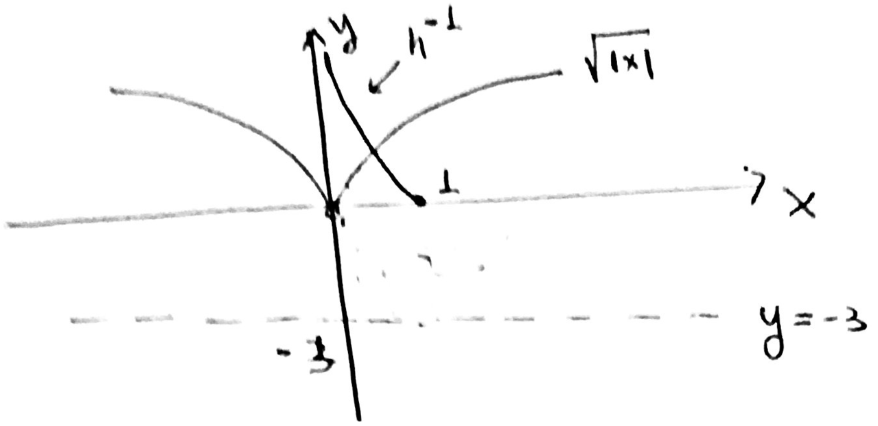
- $f''(x) = \dfrac{-6(x^2+3)^2 + 6x \cdot 2(x^2+3) \cdot 2x}{(x^2+3)^2}$

$$= \frac{-6(x^2+3)^{\cancel{2}} + 6 \cdot 4x^2 \cancel{(x^2+3)}}{(x^2+3)^{\cancel{2}}} = \frac{-6x^2 - 18 + 24x^2}{x^2+3}$$

$$= \frac{18x^2 - 18}{x^2+3} = \frac{18}{x^2+3}(x^2-1)$$

x		-1		1	
$f''(x)$	$+$	ϕ	$-$	ϕ	$+$

Β4



ΘΕΜΑ Γ

Γ1 $\lim\limits_{x\to 0}\left(-\frac{7}{2}x+\frac{1-\sigma\upsilon\nu x}{x}+1\right)=0+0+1=1$

$f(0)=1 \Leftrightarrow \alpha=1$ οπότε $f(x)=\begin{cases}-\frac{7}{2}x+\frac{1-\sigma\upsilon\nu x}{x}+1, & x<0\\ x^2+4x+1, & x\geq 0\end{cases}$

Γ2 Για $x<0$: $f'(x)=-\frac{7}{2}+\frac{\eta\mu x\cdot x+\sigma\upsilon\nu x-1}{x^2}$

Για $x>0$: $f'(x)=2x+4$

• $\lim\limits_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0^+}\frac{x^2+4x}{x}=\lim\limits_{x\to 0^+}(x+4)=4$

• $\lim\limits_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0^-}\frac{\frac{7}{2}x+\frac{1-\sigma\upsilon\nu x}{x}}{x}=\lim\limits_{x\to 0^-}\left(\frac{7}{2}+\frac{1-\sigma\upsilon\nu x}{x^2}\right)$

$\overset{*}{=}\frac{7}{2}+\frac{1}{2}=\frac{8}{2}=4$

$*\ \lim\limits_{x\to 0}\frac{1-\sigma\upsilon\nu x}{x^2}\overset{\frac{0}{0}}{=}\lim\limits_{x\to 0}\frac{(1-\sigma\upsilon\nu x)(1+\sigma\upsilon\nu x)}{x^2(1+\sigma\upsilon\nu x)}=\lim\limits_{x\to 0}\frac{\eta\mu^2 x}{x^2(1+\sigma\upsilon\nu^2 x)}$

$=\lim\limits_{x\to 0}\left(\frac{\eta\mu x}{x}\right)^2\cdot\frac{1}{1+\sigma\upsilon\nu^2 x}=1^2\cdot\frac{1}{2}=\frac{1}{2}$

Άρα, $f'(x)=\begin{cases}7/2+\frac{x\cdot\eta\mu x+\sigma\upsilon\nu x-1}{x^2}, & x<0\\ 2x+4, & x\geq 0\end{cases}$

$\boxed{\Gamma 3}$ Για $x \geq 0$: $f'(x) = 2x + 4 > 0$

άρα $f \uparrow$ στο $[0, +\infty)$ οπότε κ' "1-1"

▸ Θέτουμε $f(x) = y \Leftrightarrow x^2 + 4x + 1 = y \Leftrightarrow$

$x^2 + 4x = y - 1 \Leftrightarrow x^2 + 4x + 4 = y + 3$

$\Leftrightarrow (x+2)^2 = y + 3 \quad (y \geq -3)$

$\Leftrightarrow |x+2| = \sqrt{y+3} \underset{x \geq 0}{\Longleftrightarrow} x + 2 = \sqrt{y+3} \Leftrightarrow x = \sqrt{y+3} - 2$

Επίσης, $x \geq 0 \Leftrightarrow \sqrt{y+3} \geq 2 \Leftrightarrow y + 3 \geq 4 \Leftrightarrow y \geq 1$

οπότε $f^{-1}(x) = \sqrt{x+3} - 2, \quad x \geq 1$

$\boxed{\Gamma 4}$ ▸ $x^2 - x + 1$

$\Delta = 1 - 4 = -3 < 0$ άρα $x^2 - x + 1 > 0$ (ΑΡΑ Ο ΚΑΤΩ ΚΛΑΔΟΣ)

▸ $f^2(x^2 - x + 1) - 36 < 0 \Leftrightarrow \left(f(x^2 - x + 1) - 6\right)\left(f(x^2 - x + 1) + 6\right) < 0$

Για $x \geq 0 \Rightarrow x^2 + 4x + 1 > 0 \Leftrightarrow f(x) > 0 \ \forall x \geq 0$

άρα $f(x^2 - x + 1) + 6 > 0$.

▸ $\left(f(x^2 - x + 1) - 6\right)\underbrace{\left(f(x^2 - x + 1) + 6\right)}_{(+)} < 0 \Leftrightarrow f(x^2 - x + 1) < 6 \Leftrightarrow$

$f(x^2 - x + 1) < f(1) \underset{\substack{f \uparrow \\ x \geq 0}}{\Longleftrightarrow} x^2 - x + 1 < 1 \Leftrightarrow x^2 - x < 0$

	0		1	
+	ϕ	−	ϕ	+

άρα $x \in (0, 1)$

Θέμα Δ

Δ1 $f'(x) = \dots = \dfrac{3x^2(x-1)^2}{(3x^2-3x+1)^2} \geq 0$ άρα $f \uparrow$ στο $\mathbb{R}$.

Δ2 $$f(x) + f(1-x) = \frac{x^3}{3x^2-3x+1} + \frac{(1-x)^3}{3(1-x)^3-3(1-x)+1}$$

$$= \dots = \frac{3x^2-3x+1}{3x^2-3x+1} = 1$$

Δ3 • $g'(x) = \dfrac{e^x x - e^x}{x^2} = \dfrac{e^x(x-1)}{x^2} = \dfrac{e^x}{x^2}(x-1) > 0$ άρα

$g \uparrow$ στο $(0,1)$ οπότε κ' "1-1"

• $$\lim_{x\to-\infty} \frac{x f(g(x))}{e^x} = \lim_{x\to-\infty} \frac{f(g(x))}{\frac{e^x}{x}} = \lim_{x\to-\infty} \frac{f(g(x))}{g(x)}$$

θέτουμε $g(x) = u$, $\lim\limits_{x\to-\infty} \dfrac{e^x}{x} = \lim\limits_{x\to-\infty} e^x \cdot \dfrac{1}{x} = 0 \cdot 0 = 0$

οπότε $\lim\limits_{u\to 0} \dfrac{f(u)}{u} = \lim\limits_{u\to 0} \dfrac{u^{\cancel{3}}}{\cancel{u}(3u^2-3u+1)} = \lim\limits_{u\to 0} \dfrac{u^2}{3u^2-3u+1} = 0$

Δ4 Για $x \in \left(0, \frac{\pi}{2}\right)$:

$f(\eta\mu^2 x) + f(\sigma\upsilon\nu^2 x) = f\left(\varepsilon\varphi x \cdot e^{\sigma\upsilon\nu x - \eta\mu x}\right) \overset{\text{Δ2}}{\Leftrightarrow}$

$f(\eta\mu^2 x) + f(1-\eta\mu^2 x) = f\left(\dfrac{\eta\mu x}{\sigma\upsilon\nu x} \cdot \dfrac{e^{\sigma\upsilon\nu x}}{e^{\eta\mu x}}\right) \Leftrightarrow$

$1 = f\left(\dfrac{e^{\sigma\upsilon\nu x}}{\sigma\upsilon\nu x} \cdot \dfrac{\eta\mu x}{e^{\eta\mu x}}\right) \Leftrightarrow f(1) = f\left(\dfrac{e^{\sigma\upsilon\nu x}}{\sigma\upsilon\nu x} \cdot \dfrac{\eta\mu x}{e^{\eta\mu x}}\right) \overset{f\ 1-1}{\Leftrightarrow}$

$1 = \dfrac{e^{\sigma\upsilon\nu x}}{\sigma\upsilon\nu x} \cdot \dfrac{\eta\mu x}{e^{\eta\mu x}} \Leftrightarrow \dfrac{e^{\eta\mu x}}{\eta\mu x} = \dfrac{e^{\sigma\upsilon\nu x}}{\sigma\upsilon\nu x} \Leftrightarrow g(\eta\mu x) = g(\sigma\upsilon\nu x) \underset{(0,1)}{\overset{**\ 1-1}{\Leftrightarrow}}$

$\eta\mu x = \sigma\upsilon\nu x \Leftrightarrow \varepsilon\varphi x = \varepsilon\varphi \frac{\pi}{4} \Leftrightarrow x = \frac{\pi}{4}$ αφού $x \in \left(0, \frac{\pi}{2}\right)$

** $0 < \eta\mu x < 1$ κ' $0 < \sigma\upsilon\nu x < 1$ επειδή $x \in \left(0, \frac{\pi}{2}\right)$