A1

1.) $\lim\limits_{x \to -4} f(x) = 2$

2.) $\lim\limits_{x \to -3} f(x) = 0$

3.) $\lim\limits_{x \to -2} f(x)$ Δεν υπάρχει

4.) $\lim\limits_{x \to -1} f(x) = 2$

5.) $\lim\limits_{x \to 0} f(x) = 2$

6.) $\lim\limits_{x \to 2} f(x) = -2$

7.) $\lim\limits_{x \to 3} f(x)$ Δεν υπάρχει

8.) $\lim\limits_{x \to 4} f(x) = 5$

9.) $\lim\limits_{x \to 6} f(x) = 1$

A2) 1.

$$\lim_{x \to 2} \frac{|x^3 - x - 1| - |x - 7|}{x^2 - 4} = \lim_{x \to 2} \frac{x^3 - \cancel{x} - 1 + \cancel{x} - 7}{x^2 - 4} = \lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4} =$$

$$\left| \begin{array}{l} x^3 - x - 1 > 0 \text{ κοντά στο } 2 \\ x - 7 < 0 \text{ κοντά στο } 2 \end{array} \right|$$

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x^2 + 2x + 4)}{\cancel{(x-2)}(x+2)} = \frac{2^2 + 2 \cdot 2 + 4}{4} = \frac{12}{4} = 3.$$

5.

$$\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{x^2 - 1} \underset{\substack{\text{όταν} \\ x \to 1 \\ u \to 1}}{\overset{\text{θέτουμε}}{\underset{u = \sqrt[3]{x}}{=}}} \lim_{u \to 1} \frac{u - 1}{u^6 - 1} = \lim_{u \to 1} \frac{u - 1}{(u^3 - 1)(u^3 + 1)} =$$

$$= \lim_{u \to 1} \frac{\cancel{u - 1} \; 1}{\cancel{(u-1)}(u^2 + u + 1)(u^3 + 1)} = \frac{1}{3 \cdot 2} = \frac{1}{6}.$$

2.)

$$\lim_{x \to 3} \frac{\sqrt{x^2+7} - 4}{x - 3} = \lim_{x \to 3} \frac{(\sqrt{x^2+7} - 4)(\sqrt{x^2+7} + 4)}{(x-3)(\sqrt{x^2+7} + 4)} =$$

$$= \lim_{x \to 3} \frac{\sqrt{x^2+7}^{\,2} - 16}{(x-3)(\sqrt{x^2+7} + 4)} = \lim_{x \to 3} \frac{x^2 - 9}{(x-3)(\sqrt{x^2+7} + 4)} =$$

$$= \lim_{x \to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)} \cdot (\sqrt{x^2+7} + 4)} = \frac{6}{\sqrt{16} + 4} = \frac{6}{4+4} = \frac{6}{8} = \frac{3}{4}.$$

3.) $\lim\limits_{x \to 1} \dfrac{x^3+4x^2-3x-2}{x^2-3x+2} = \dfrac{\cancel{(x-1)} \cdot (x^2+5x+2)}{\cancel{(x-1)} \cdot (x-2)} = \dfrac{8}{-1} = -8$

1	4	-3	-2	$\rho=1$
$\downarrow$	1	5	$+2$	$(x-1):0$
1	5	2	$\boxed{0}$	παράγοντας

$(x-1) \cdot (x^2+5x+2)$

x^2-3x+2, $\Delta = (-3)^2 - 4 \cdot 1 \cdot 2 = 9-8 = 1$, $x_1, x_2 = \dfrac{3 \pm 1}{2}$ $\begin{cases} x_1 = 2 \\ x_2 = 1 \end{cases}$

4.) $\lim\limits_{x \to 0} \dfrac{\eta\mu x}{\sqrt{x+1}-1} = \lim\limits_{x \to 0} \dfrac{\eta\mu x \cdot (\sqrt{x+1}+1)}{(\sqrt{x+1}-1)(\sqrt{x+1}+1)} =$

$= \lim\limits_{x \to 0} \dfrac{\eta\mu x \cdot (\sqrt{x+1}+1)}{\sqrt{x+1}^{\,2} - 1} = \lim\limits_{x \to 0} \dfrac{\eta\mu x}{x} \cdot (\sqrt{x+1}+1) = 1 \cdot 2 = 2.$

A3.) 1.) $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \dfrac{-\eta\mu x}{x^2-x} = \lim\limits_{x \to 0^-} \dfrac{-\eta\mu x}{x(x-1)} = \lim\limits_{x \to 0^-} \dfrac{\eta\mu x}{x} \cdot \dfrac{-1}{x-1} = (+1)(-1) = +1$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \dfrac{x^3-1}{x-1} = \dfrac{-1}{-1} = 1$, συνεπώς, $\lim\limits_{x \to 1} f(x) = 1$

2.) $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} \dfrac{\cancel{(x-1)}(x^2+x+1)}{\cancel{x-1}} = 1+1+1 = 3$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \dfrac{x^2-3x+2}{x^2+2x-3} = \lim\limits_{x \to 1^+} \dfrac{\cancel{(x-1)}(x-2)}{\cancel{(x-1)}(x+3)} = \dfrac{1-2}{1+3} = -\dfrac{1}{4}$

Συνεπώς το όριο <u>δεν</u> υπάρχει.

B1.) $4x\sqrt{x^2+3} \le (x-1)f(x) + 8x \le 5x^2+3$

$4x\sqrt{x^2+3} - 8x \le (x-1) \cdot f(x) \le 5x^2 - 8x + 3$

1.) · Διαιρούμε με $x-1<0$, δηλ. $(x<1)$

$\dfrac{4x\sqrt{x^2+3}-8x}{x-1} \ge f(x) \ge \dfrac{5x^2-8x+3}{x-1}$

$\lim\limits_{x \to 1^-} \dfrac{4x\sqrt{x^2+3}-8x}{x-1} = 4\lim \dfrac{x(\sqrt{x^2+3}-2)(\sqrt{x^2+3}+2)}{(x-1)(\sqrt{x^2+3}+2)} =$

$= 4\lim\limits_{x \to 1^-} \dfrac{x(\sqrt{x^2+3}^{\,2}-4)}{(x-1)(\sqrt{x^2+3}+2)} = 4\lim\limits_{x \to 1} \dfrac{x(x^2-1)}{(x-1)(\sqrt{x^2+3}+2)} =$

$$= 4\lim_{x\to 1^-}\frac{x\cdot\cancel{(x-1)}\cdot(x+1)}{\cancel{(x-1)}(\sqrt{x^2+3}+2)} = 4\lim\frac{1\cdot 2}{2+2} = \cancel{4}\cdot\frac{2}{\cancel{4}} = 2.$$

$$\lim_{x\to 1^-}\frac{5x^2-8x+3}{x-1} = \lim_{x\to 1^-}\frac{5x^2-5x-3x+3}{x-1} = \lim_{x\to 1^-}\frac{5x(x-1)-3(x-1)}{x-1} =$$

$$= \lim_{x\to 1^-}\frac{\cancel{(x-1)}(5x-3)}{\cancel{x-1}} = 5-3 = 2.$$

Άρα, $\lim\limits_{x\to 1^-} f(x) = 2$ από κριτ. παρεμβολής.

Ομοίως, διαιρώντας με $x-1>0$ (δηλ. $x>1$)

$$\frac{4x\sqrt{x^2+3}-8x}{x-1} \le f(x) \le \frac{5x^2-8x+3}{x-1}$$

$$\left.\begin{array}{l} \lim\limits_{x\to 1^+}\dfrac{4x\sqrt{x^2+3}-8x}{x-1} = 2 \\ \text{και} \\ \lim\limits_{x\to 1^+}\dfrac{5x^2-8x+3}{x-1} = 2 \end{array}\right\}$$ Άρα, $\lim\limits_{x\to 1^+} f(x) = 2$ από κριτ. παρεμβολής

Συνεπώς, $\lim\limits_{x\to 1} f(x) = 2$.

2.) $$\lim_{x\to 1}\frac{(x-1)\cdot f(x)+\eta\mu(\pi\cdot x)}{x^2-3x+2} = \lim_{x\to 1}\frac{(x-1)f(x)+\eta\mu(\pi\cdot x)}{(x-1)(x-2)} =$$

$$= \lim_{x\to 1}\frac{f(x)+\frac{\eta\mu(\pi x)}{x-1}}{(x-2)} = A$$

$$L = \lim_{x\to 1}\frac{\eta\mu\pi x}{x-1} \xrightarrow[\text{όταν } x\to 1,\ u\to 0]{x-1=u} \lim_{u\to 0}\frac{\eta\mu(u+1)\cdot\pi}{u} = \lim_{u\to 0}\frac{\eta\mu(\pi u+\pi)}{u} =$$

$$= \lim_{u\to 0}\frac{\eta\mu\pi u\cdot\sigma\upsilon\nu\pi + \sigma\upsilon\nu\pi u\,\eta\mu\pi}{u} = -\lim_{u\to 0}\frac{\eta\mu\pi u}{u} = -\pi\cdot\lim_{u\to 0}\frac{\eta\mu\pi u}{\pi u} = -\pi$$

Άρα, $$A = \lim_{x\to 1}\frac{f(x)+\frac{\eta\mu\pi x}{x-1}}{x-2} = \frac{2-\pi}{1-2} = \pi-2.$$

B2.) $\lim\limits_{x\to0}\frac{f(x)}{x\cdot ημx}=3$, θέτουμε $r(x)=\frac{f(x)}{x\cdot ημx}$ με $\lim\limits_{x\to0}r(x)=3$

$$\boxed{x\cdot ημx\cdot r(x)=f(x)}$$

$\lim\limits_{x\to0}\left[\left(\sqrt{x^2+1}-1\right)\cdot g(x)\right]=5$, θέτουμε $\ell(x)=g(x)\cdot\left(\sqrt{x^2+1}-1\right)$ με $\lim\limits_{x\to0}\ell(x)=5$

$$g(x)=\frac{\ell(x)}{\sqrt{x^2+1}-1}$$

1.) $$\lim_{x\to0}[f(x)\cdot g(x)]=\lim_{x\to0}\frac{x\cdot ημx\cdot r(x)\cdot\ell(x)}{\sqrt{x^2+1}-1}=\lim_{x\to0}\frac{x\,ημx\,r(x)\,\ell(x)\left(\sqrt{x^2+1}+1\right)}{\left(\sqrt{x^2+1}-1\right)\left(\sqrt{x^2+1}+1\right)}$$

$$=\lim_{x\to0}\frac{x\cdot ημx\cdot r(x)\cdot\ell(x)\left(\sqrt{x^2+1}+1\right)}{\sqrt{x^2+1}^{\,2}-1^2}=\lim_{x\to0}\frac{\cancel{x}\,ημx\cdot r(x)\,\ell(x)\left(\sqrt{x^2+1}+1\right)}{x^{\cancel{2}}}$$

$$=\lim_{x\to0}\left[\frac{ημx}{x}\cdot r(x)\cdot\ell(x)\cdot\left(\sqrt{x^2+1}+1\right)\right]=1\cdot3\cdot5\cdot2=30.$$

2.) $$\lim_{x\to0}\frac{2f(x)-x^4\cdot g(x)}{f(x)+ημ^2x}=\lim_{x\to0}\frac{2+\frac{x^4g(x)}{f(x)}}{1+\frac{ημ^2x}{f(x)}}=$$

$$=\lim_{x\to0}\frac{2+\frac{x^4\cdot\ell(x)}{\left(\sqrt{x^2+1}-1\right)\cdot x\cdot ημx\cdot r(x)}}{1+\frac{ημ^{\cancel{2}}x}{x\,\cancel{ημx}\cdot r(x)}}=$$

$$=\lim_{x\to0}\frac{2+\frac{\cancel{x^4}\,\ell(x)\left(\sqrt{x^2+1}+1\right)}{\cancel{x^2}\cdot\cancel{x}\cdot ημx\cdot r(x)}}{1+\frac{ημx}{x}\cdot\frac{1}{r(x)}}=\frac{2-\frac{10}{3}}{1+1\cdot\frac{1}{3}}=$$

$$=\frac{\frac{6}{3}-\frac{10}{3}}{1+\frac{1}{3}}=\frac{-\frac{4}{3}}{\frac{4}{3}}=-1.$$

ΘΕΜΑ Γ)

Γ1.) 1.) $\frac{1}{2}x^3 + \frac{17}{6}x^2 + 3x - \frac{4}{3} = 0$

$\cancel{6}^{3} \cdot \frac{1}{\cancel{2}}x^3 + \cancel{6} \cdot \frac{17}{\cancel{6}}x^2 + 6 \cdot 3x - \cancel{6}^{2} \cdot \frac{4}{\cancel{3}} = 0$

$3x^3 + 17x^2 + 18x - 8 = 0$

Θέτουμε $P(x) = 3x^3 + 17x^2 + 18x - 8$

$P(-2) = 3 \cdot (-2)^3 + 17 \cdot (-2)^2 + 18 \cdot (-2) - 8$

$= 3 \cdot (-8) + 17 \cdot 4 - 36 - 8$

$= -24 + 68 - 44 = 0$

Άρα, το -2 ρίζα

3	17	18	-8	$\rho = -2$
↓	-6	-22	+8	$(x+2)$: ο παράγοντας
3	11	-4	$\boxed{0}$	

$P(x) = (x+2) \cdot (3x^2 + 11x - 4)$

$\Delta = 11^2 - 4 \cdot 3 \cdot (-4) = 121 + 48 = 169$

$x_1, x_2 = \frac{-11 \pm 13}{6}$ ↗ $x_1 = \frac{2}{6} = \frac{1}{3}$, ↘ $x_2 = \frac{-24}{6} = -4$

Άρα, ρίζες της εξίσωσης -4, $\frac{1}{3}$ και -2.

Γ2.) 2) $x^2 + 4 - \frac{7x+10}{x+2} = 2x + \frac{4}{x+2}$, θα πρέπει: $x + 2 \neq 0$ $\boxed{x \neq -2}$

$x^2(x+2) + 4(x+2) - \frac{7x+10}{\cancel{x+2}} \cdot \cancel{(x+2)} = 2x(x+2) + \frac{4}{\cancel{x+2}} \cdot \cancel{(x+2)}$

$x^3 + \cancel{2x^2} + 4x + 8 - (7x+10) = \cancel{2x^2} + 4x + 4$

$x^3 + \cancel{4x} + 8 - 7x - 10 = \cancel{4x} + 4$

$x^3 - 7x - 6 = 0$, θέτουμε $P(x) = x^3 - 7x - 6$

$P(-1) = (-1)^3 - 7 \cdot (-1) - 6$

$= -1 + 7 - 6 = 0$

άρα το -1 ρίζα του $P(x)$

1	0	-7	-6	$\rho = -1$
↓	-1	1	+6	$(x+1)$: ο παράγοντας
1	-1	-6	0	

$P(x) = (x+1) \cdot (x^2 - x - 6)$

$x^2 - x - 6$, $\Delta = (-1)^2 - 4 \cdot 1 \cdot (-6) = 1 + 24 = 25$

$x_1, x_2 = \frac{+1 \pm 5}{2} = \begin{cases} x_1 = 3 & \text{Δεκτή} \\ x_2 = -2 & \text{Απορ.} \end{cases}$

Άρα, λύσεις της εξίσωσης 3 και -1.

Γ1.)3.) $\sqrt{x+8} = 2 + \sqrt{x-4}$

Θα πρέπει

$\left.\begin{array}{l} x+8 \geq 0 \\ \text{και} \\ x-4 \geq 0 \end{array}\right\} \begin{array}{l} x \geq -8 \\ \\ x \geq 4 \end{array} \Bigg\} \boxed{x \geq 4}$

και

$2 + \sqrt{x-4} \geq 0$ που ισχύει για κάθε $x \in \mathbb{R}$

$\sqrt{x+8}^{\,2} = (2 + \sqrt{x-4})^2$

$x + 8 = 4 + 4\sqrt{x-4} + \sqrt{x-4}^{\,2}$

$\cancel{x} + 8 = \cancel{4} + 4\sqrt{x-4} + \cancel{x} - \cancel{4}$

$8 = 4\sqrt{x-4}$

$2 = \sqrt{x-4}$

$2^2 = \sqrt{x-4}^{\,2}$

$4 = x - 4$

$x = 8$ Δεκτή

Γ2.)1.) $-x^3 + 3x + 2 < 0$

Θέτουμε $P(x) = -x^3 + 3x + 2$

$P(-1) = -(-1)^3 + 3(-1) + 2 = +1 - 3 + 2 = 0$

Άρα, το -1 ρίζα του $P(x)$

-1	0	3	2	$\rho = -1$
↓	1	-1	-2	
-1	1	2	$\boxed{0}$	

$(x+1)$: ο παράγοντας

$P(x) = (x+1)(-x^2 + x + 2)$

$\Delta = 1^2 - 4 \cdot (-1) \cdot 2 =$
$= 1 + 8 = 9$

$x_1, x_2 = \frac{-1 \pm 3}{-2} = \begin{cases} x_1 = -1 \\ x_2 = +2 \end{cases}$

x	$-\infty$	-1		2	$+\infty$
$x+1$	$-$	0	$+$		$+$
$-x^2+x+2$	$-$	0	$+$	0	$-$
$P(x)$	$+$	0	$+$	0	$-$

$x \in (2, +\infty)$

Γ2.) 2.) $\dfrac{x^2-6x-7}{x-4} \leq 0 \iff (x^2-6x-7)(x-4) \leq 0$ με $x-4 \neq 0$ $\boxed{x \neq 4}$

$\Delta = (-6)^2 - 4 \cdot 1 \cdot (-7)$
$= 36 + 28 = 64$

$x_1, x_2 = \dfrac{6 \pm 8}{2}$, $x_1 = 7$, $x_2 = -1$

x	$-\infty$	-1		4		7	$+\infty$
$x-4$	$-$		$-$	0	$+$		$+$
x^2-6x-7	$+$	0	$-$		$-$	0	$+$
P	$-$	0	$+$	‖	$-$	0	$+$

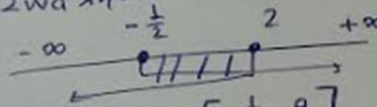
$x \in (-\infty, -1] \cup (4, 7]$

Γ2.) 3.) $\sqrt{2x+1} - \sqrt{x+3} \leq 0$ θα πρέπει: $2x+1 \geq 0 \iff x \geq -\frac{1}{2}$
και
$x+3 \geq 0 \iff x \geq -3$

Δηλ. $\boxed{x \geq -\frac{1}{2}}$

$\sqrt{2x+1} \leq \sqrt{x+3}$
$\sqrt{2x+1}^2 \leq \sqrt{x+3}^2$
$2x+1 \leq x+3$
$\boxed{x \leq 2}$

Συναληθεύοντας, έχουμε



$x \in [-\frac{1}{2}, 2]$

Γ2.) 4.) $\dfrac{x^3+6x-5}{x^2-9} + \dfrac{3x}{x+3} + \dfrac{5}{x-3} + 2 \geq 0$, θα πρέπει
$x+3 \neq 0$
και
$x-3 \neq 0$
και
$x^2-9 \neq 0$
$\boxed{x \neq 3 \text{ και } x \neq -3}$

$\dfrac{x^3+6x-5}{(x-3)(x+3)} + \overset{x-3}{\dfrac{3x}{x+3}} + \overset{x+3}{\dfrac{5}{x-3}} + \overset{(x-3)(x+3)}{\dfrac{2}{1}} \geq 0$

$\dfrac{x^3+6x-5+3x(x-3)+5(x+3)+2(x-3)(x+3)}{(x-3)(x+3)} \geq 0$

$\dfrac{x^3+6x-5+3x^2-9x+5x+15+2(x^2-9)}{(x-3)(x+3)} \geq 0$

$\dfrac{x^3+3x^2+2x+10+2x^2-18}{(x-3)(x+3)} \geq 0 \iff \dfrac{x^3+5x^2+2x-8}{(x-3)\cdot(x+3)} \geq 0$

Έστω $h(x) = x^3+5x^2+2x-8$ με $h(1) = 1+5+2-8 = 0$ σελ. 7
1 ρίζα του $h(x)$

1	5	2	-8	$p=1$
↓	1	6	8	$(x-1)$: 0
1	6	8	$\boxed{0}$	παράγοντας

$h(x)=(x-1)\cdot(x^2+6x+8)$

$\Delta=6^2-4\cdot1\cdot8$
$=36-32=4$

$x_1,x_2=\dfrac{-6\pm2}{2}$ → $x_1=-2$, $x_2=-4$

$$\frac{(x-1)\cdot(x^2+6x+8)}{(x-3)\cdot(x+3)}\geq 0 \Leftrightarrow (x-1)\cdot(x^2+6x+8)\cdot(x-3)(x+3)\geq 0$$

με $x\in\mathbb{R}-\{-3,3\}$

x	$-\infty$		-4		-3		-2		1		3		$+\infty$
$x-1$		$-$		$-$	‖	$-$		$-$	0	$+$	‖	$+$	
x^2+6x+8		$+$	0	$-$	‖	$-$	0	$+$		$+$	‖	$+$	
$x-3$		$-$		$-$	‖	$-$		$-$		$-$	0	$+$	
$x+3$		$-$		$-$	0	$+$		$+$		$+$	‖	$+$	
P		$-$	0	$+$	‖	$-$	0	$+$	0	$-$	‖	$+$	

$x\in[-4,-3)\cup[-2,1]\cup(3,+\infty)$

ΘΕΜΑ Δ 1) $P(x)=x^3-(3a-3)x+2a-2$

$P(1)=0$

$1^3-(3a-3)\cdot1+2a-2=0$

$1-3a+3+2a-2=0$

$-a+2=0$

$\boxed{a=2}$, για $a=2$: $P(x)=x^3-3x+2$

2.) $P(x)=0$

$x^3-3x+2=0$, $P(1)=0$, (προηγ. ερωτ.)

1	0	-3	2	$p=1$
↓	1	1	-2	$(x-1)$: 0 παράγοντας.
1	1	-2	$\boxed{0}$	

$P(x)=(x-1)\cdot(x^2+x-2)$

$\Delta=1^2-4\cdot1\cdot(-2)=1+8=9$, $x_1,x_2=\dfrac{-1\pm3}{2}$ → $x_1=1$, $x_2=-2$

Άρα, $P(x)=0$, διπλή ρίζα το 1
και ρίζα το 2.

3.) $x^3 \quad -3x+2 \;|\; x^2-5x-14$

$-x^3+5x^2+14x \;|\; x+5$

$5x^2+11x+2$

$-5x^2+25x+70$

$36x+72$

$P(x) = (x^2-5x-14)\cdot(x+5)+36x+72$

4.) $P(x) \le 36x+72$

$(x^2-5x-14)(x+5)+\cancel{36x}+\cancel{72} \le \cancel{36x}+\cancel{72}$

$(x^2-5x-14)(x+5) \le 0$

$\Delta = (-5)^2-4\cdot 1\cdot(-14) = 25+56 = 81$

$x_1, x_2 = \frac{5 \pm 9}{2}$ ↗ $x_1=7$, ↘ $x_2=-2$

x	$-\infty$ -5	-2	7	$+\infty$
$x^2-5x-14$	+ 0	−	− 0	+
$x+5$	−	− 0	+	+
γ	− 0	+ 0	− 0	+

$x \in (-\infty, -5] \cup [-2, 7]$

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5) $P(x-1) = (x-1)^3-3(x-1)+2 = x^3-3x^2+\cancel{3x}-1-\cancel{3x}+3+2 =$

$= x^3-3x^2+4$

$Q(x) = \frac{x^3-3x^2+\cancel{4}-\cancel{4}}{x+2} = \frac{x^2(x-3)}{x+2}$

Για να είναι η C_Q πάνω από τον $x'x$ λύνουμε

$Q(x) > 0$

$\frac{x^2(x-3)}{x+2} > 0 \Leftrightarrow x^2(x-3)(x+2) > 0$ με $x+2 \ne 0 \Leftrightarrow x \ne -2$

x	$-\infty$ -2	3	$+\infty$
x^2	+	+	+
$x-3$	−	− 0	+
$x+2$	−	+	+
$Q(x)$	+	− 0	+

Άρα, για $x \in (-\infty, -2) \cup (3, +\infty)$